GENERIC PROPERTIES OF COUNTABLY INFINITE GROUPS

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SETTING

FIXED COMMON UNIVERSE: $\mathbb{N}=\{1,2,\dots\}$

$\mathcal{G}=\{G\in\mathbb{N}^{\mathbb{N}\times\mathbb{N}} : G$ IS A GROUP OPERATION AND 1 IS ITS IDENTITY$\}$

$\mathcal{G}\subseteq \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ G_δ $\Rightarrow$ POLISH

<u>REMARK:</u> $S_\infty^* = \{\sigma\in S_\infty : \sigma(1)=1\}$

$S_\infty^* \curvearrowright \mathcal{G}$ CONTINUOUSLY: $\sigma.G$ = PUSH-FORWARD OF G ALONG σ

- ORBITS = ISOM. CLASSES
- THIS ACTION MAKES $\mathcal{G}$ NICE!
- THE BASIC CLOPEN SETS:

*	2	3	*		
*	*	*	*		
*	*	5	*		
		2			

<u>DEF:</u> A <u>GROUP PROPERTY</u> IS AN ISOM-INVARIANT SUBSET OF $\mathcal{G}$.

QUESTIONS

<u>MAIN QUESTION</u>: WHAT PROPERTIES ARE GENERIC IN $\mathcal{G}$?

(IN THE SENSE OF BAIRE CATEGORY)

SPECIFIC QUESTIONS:

(A) COMEAGER ISOM. CLASS?

(B) GENERIC SUBGROUPS?

(C) INTERESTING SUBSPACES?

(D) CONNECTIONS TO OTHER SETTINGS?

TWO NOTIONS FROM GROUP THEORY

DEF: A FINITELY GENERATED GROUP G HAS SOLVABLE WORD PROBLEM IF THERE EXISTS A TURING MACHINE THAT DECIDES FOR EVERY WORD IN THE GENERATORS OF G WHETHER IT REPRESENTS THE IDENTITY OR NOT.

DEF: THE GROUP G IS ALGEBRAICALLY CLOSED IF FOR EVERY FINITE SYSTEM Γ OF EQUATIONS AND INEQUATIONS WITH PARAMATERS FROM G WE HAVE:

$$(\exists H \geq G \quad \Gamma \text{ IS SOLVABLE IN } H) \Rightarrow (\Gamma \text{ IS SOLVABLE IN } G)$$

EX:

$$\begin{cases} g_1 x_1 g_2^{-3} x_4^2 = 1 \\ g_2 x_2 = 1 \\ x_2^{-1} g_3 x_1 \neq 1 \end{cases}$$

THM: (GKL) ALG. CLOSEDNESS IS GENERIC

COR: ALSO GENERIC: NOT LOC. FIN., NOT FIN. GEN., SIMPLE, INNER ULTRA HOMOGENEOUS

GENERIC SUBGROUPS (B)

<u>DEF:</u> THE GROUP H IS <u>GENERICALLY EMBEDDABLE</u> IF THE SET

$\mathcal{E}_H = \{ G \in \mathcal{G} : H \text{ EMBEDS INTO } G \}$ IS COMEAGER.

<u>THM:</u> (EGKKK) TFAE:

(1) H IS GENERICALLY EMBEDDABLE

(2) H EMBEDS INTO EVERY ALG. CLOSED GROUP

(3) H IS CTBL AND EVERY FINITELY GEN. SUBGROUP OF H HAS SOLVABLE WORD PROBLEM.

<u>REMARKS:</u> (2) $\Leftrightarrow$ (3) : B.H. NEUMANN, H. SIMMONS, A. MACINTYRE (3 PAPERS!)

(2) $\Rightarrow$ (1) : FOLLOWS FROM THE FACT THAT ALG. CLOSEDNESS IS GENERIC.

<u>PROOF</u>: (1) $\Rightarrow$ (2) (SKETCH)

(1) H IS GENERICALLY EMBEDDABLE

(2) H EMBEDS INTO EVERY ALG. CLOSED GROUP

- ENOUGH: FOR FIN. GEN. H (BY HOMOGENEITY)
- FIX: $\underline{\underline{G_0}}$ ALG. CLOSED, $\underline{\underline{H}}$ FIN. GEN. GEN. EMB.
- $\mathcal{E}_H$ IS G_δ AND COMEAGER $\Rightarrow$ $\operatorname{int} \mathcal{E}_H \neq \emptyset$
- THE ISOM. CLASS OF G_0 IS DENSE

$\Rightarrow$ ✓ □

ISOMORPHISM CLASSES (A)

THM: (GKL) EVERY ISOM. CLASS IS MEAGER IN $\mathcal{G}$.

PROOF: (SKETCH)

$\lightning$ SUPPOSE: THE ISOM. CLASS OF G IS COMEAGER.

$\Rightarrow$ (H GEN. EMB $\Leftrightarrow$ $H \leq G$)

- G IS ALG. CLOSED
  +
  MILLER'S THM

$\Rightarrow \exists H \leq G$ FIN. GEN. WITH UNSOLVABLE WORD PROBLEM $\lightning$ $\square$

SUBSPACES (C)

NOW: ABELIAN GROUPS, FOR MORE: GKL

THE SUBSPACE $\mathcal{A} \subseteq \mathcal{G}$ OF ABELIAN GROUPS IS CLOSED IN $\mathcal{G}$

THM: (EGKKK) $\exists$ COMEAGER ISOM. CLASS IN $\mathcal{A}$:

$$A = \bigoplus_{p \text{ PRIME}} \bigoplus_{i \in \mathbb{N}} \underset{\substack{\uparrow \\ \text{PRÜFER } p\text{-GROUP}}}{\mathbb{Z}[p^\infty]} = \bigoplus_{i \in \mathbb{N}} \left(\mathbb{Q}/\mathbb{Z} \right)$$

REMARK: A IS THE UNIVERSAL CTBL ABELIAN TORSION GROUP. ALSO FRAÏSSÉ LIMIT OF FINITE ABELIAN GROUPS

PROOF: (SKETCH) 4 FOUR PROPERTIES CHARACTERIZE A:

- CTBL ABELIAN
- TORSION
- DIVISIBLE
- $\forall$ FIN. AB. GROUP EMBEDS INTO A

$\Big\}$ DENSE + G_δ ONE-BY-ONE

$\square$

OTHER SETTINGS (D)